

ON THE POLAR DISTRIBUTION FOR SINGULARITIES EQUATIONS OF NONLINEAR PDES

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ABSTRACT. In the beginning of this note we review the fundamentals of Grassmann and flag bundles, stressing their key role in the geometrical formalism of nonlinear PDEs. Then we observe that the equation of involutive planes of the Cartan distribution—a nonlinear PDE naturally associated with any jet space—is equipped with a polar distribution. The equation plays the role of “empty equation” in the context of singularity equations, while the polar distribution is a source of contact invariants. In the last part, some examples of polar distributions are described explicitly.

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INTRODUCTION

The equation of involutive k -planes of the Cartan distribution ($\mathcal{I}_k(\mathcal{C})$, in this note) can be seen as a coordinate-free way of defining higher-order jet spaces out of the lower-order ones; as such, the idea behind it is very old, and it is impossible to point an adequate reference. Most of the textbooks [5, 1, 20, 16, 2, 19] on the geometry of jet spaces and nonlinear PDEs, in one form or another, exploit it; it can be also found in the neighboring area of geometric mechanics disguised as a “constraint” in the sentence *higher-order mechanics can be seen as a constrained first-order ones*. However, it has never given a special attention as an equation *per se*, since its role is not fundamental for the study of the geometry of nonlinear PDEs and their (regular) solutions.

Nonetheless, if singular solutions are brought into play, then the role of $\mathcal{I}_k(\mathcal{C})$, where k is less than the number of independent variables, becomes not so choreographic—a good reason, in the authors’ opinion, for studying it in detail (Section 3). Besides the context of singular solutions, whose main contributions come from Vinogradov, Krishchenko and Lychagin [6, 7, 9, 10, 11, 12, 13, 18, 8, 14, 3], involutive $(n-1)$ -planes are at the very heart of the theory of characteristics, since the former are, in a sense, dual to the latter. Such a relationship between singularity equations and characteristics, firstly observed by Krishchenko [7, 6] and Hermann, admits a transparent and simple formulation (Corollary 21) in terms of *involutive flags*. These are an obvious generalization of the equation of involutive planes (Definition 14), and a special cases of *integral flags*, already appearing in the theory of exterior differential systems (e.g., in the proof of the Cartan–Kähler theorem [2]).

Grassmann and flag bundles (Section 1) have been put at the basis of our analysis not for merely aesthetic reasons. Indeed, this allows to construct a *polar distribution* on $\mathcal{I}_k(\mathcal{C})$ by standard manipulations of multi-linear algebra (Section 2). We called it *polar* since, quoting Bryant&Griffiths [2], *is related to the linearization process in multi-linear algebra known as “polarization”, the most common example being the polarization of a quadratic form on a vector space to produce a bilinear form*. Examples of polar distributions (\mathbb{D} , in this note) can be found in the context of Monge–Ampère equations (to the authors knowledge, firstly by Vinogradov [4, 14, 21]) and the idea behind it already appears in Bryant&Griffiths’ classical book [2] (in the guises of *polar space* or *space of integral enlargements*), but it has never been studied in full generality. This task, which in fact poses no serious threat, is carried out here, in the framework of Grassmann bundles. The subtle point is the natural character of the construction, which, though never stressed throughout the paper, is its most important feature: it implies, in particular, that \mathbb{D} is a contact invariant of the nonlinear PDE it is associated with.

Besides Corollary 21, the main results of this note are contained in the last Section 5, where it is proved, in some cases, that \mathbb{D} is isomorphic to the Cartan distribution on a jet space with less independent variables. This fact allow, for instance, to associate with a k^{th} order PDE in 2 independent variables, a family of ODEs which is, by construction, a contact invariant of the given PDE.

Notations and conventions. All vector spaces are over \mathbb{R} . If P is a $C^\infty(M)$ -module, we replace $C^\infty(M)$ by M in all statements involving P (e.g., duals, tensor products, homomorphisms). Also P_x denotes the fiber of the bundle corresponding to P , at the point $x \in M$. The same notation is used for non-linear bundles: e.g., E_x is the fiber of $\pi : E \rightarrow M$ at $x \in M$. If $F : N \rightarrow M$ is a smooth map, $F^\circ(\pi)$ denotes the induced bundle on M . The dual of vector spaces (or module) P is denoted by P^\vee .

All manifolds and maps are assumed to be smooth and all PDEs are supposed to be formally integrable ones. Symbol FT denote the tangential map associated with F . We adopt Einstein notation for repeated indexes.

Concerning definitions and main structures of jet spaces and nonlinear PDEs, we adhere to the notation of [1].

1. PRELIMINARY RESULTS ABOUT GRASSMANNIAN BUNDLES

If V is an $(n+m)$ -dimensional vector space, $\text{Gr}(V, n)$ denotes the nm -dimensional smooth manifold made of n -dimensional linear subspaces of V . The short exact sequence of vector bundles

$$(1.1) \quad \begin{array}{ccccc} R(V, n) & \hookrightarrow & \text{Gr}(V, n) \times V & \xrightarrow{p} & N(V, n) \\ & \searrow R & \downarrow \pi_{\text{Gr}} & \swarrow N & \\ & & \text{Gr}(V, n) & & \end{array}$$

where π_{Gr} is the trivial bundle, R is the $(n$ -dimensional) *tautological* (or *canonical*) bundle, and N is the $(m$ -dimensional) *normal* bundle, is known as the *universal sequence* over $\text{Gr}(V, n)$. By definition, $R_L = L$ (hence the name “tautological”) and $N_L = \frac{V}{L}$, for all $L \in \text{Gr}(V, n)$. Together, the tautological and the normal bundle allow to describe the tangent bundle τ_{Gr} of $\text{Gr}(V, n)$, since

$$\tau_{\text{Gr}} \cong \text{Hom}(R, N) = R^\vee \otimes N$$

canonically. In order to introduce the *characteristics* of a distribution on $\text{Gr}(V, n)$, it is convenient to prove the next Corollary 1.

Corollary 1. *A projective sub-bundle τ'_{Gr} exists, such that $\mathbb{P}(\tau'_{\text{Gr}})$ is the image of the Veronese embedding*

$$\mathbb{P}(R^\vee) \times_{\text{Gr}(V, n)} \mathbb{P}(N) \subseteq \mathbb{P}(\tau_{\text{Gr}}).$$

Proof. Just observe that, thanks to (1), one can speak of the *rank* of a tangent vector to $\text{Gr}(V, n)$, which is the rank of the corresponding homomorphism from R to N . Then

$$T'_L \text{Gr}(V, n) \stackrel{\text{def}}{=} \{\theta \in T_L \text{Gr}(V, n) \mid \text{rank}(\theta) = 1\}$$

is a projective variety in $T_L \text{Gr}(V, n)$ for all $L \in \text{Gr}(V, n)$. Define τ'_{Gr} as the bundle whose fiber at L is (1). \square

Let now Δ be a distribution on $\text{Gr}(V, n)$, regarded as a vector sub-bundle $\Delta \subseteq \tau_{\text{Gr}}$. Hence, $\mathbb{P}(\Delta)$ is a smooth sub-bundle of $\mathbb{P}(\tau_{\text{Gr}})$, and it makes sense to consider the sub-bundle

$$(1.2) \quad \mathbb{P}(\tau'_{\text{Gr}}) \cap \mathbb{P}(\Delta)$$

of $\mathbb{P}(\Delta)$, made of rank-one directions belonging to Δ . Denote by $p_1 : \mathbb{P}(R^\vee) \times_{\text{Gr}(V, n)} \mathbb{P}(N) \rightarrow \mathbb{P}(R^\vee)$ the canonical projection, and take the image of (1.2) via p_1 . The so-obtained projective sub-bundle

$$\text{char}^{\mathbb{R}}(\Delta) \stackrel{\text{def}}{=} p_1(\mathbb{P}(\tau'_{\text{Gr}}) \cap \mathbb{P}(\Delta))$$

of $\mathbb{P}(R^\vee)$ is known as the *bundle of characteristics* of Δ .

In order to pass to the global point of view, one just needs to “replace” vector spaces by vector bundles. To make this rigorous, it is sufficient to observe that the universal sequence (1.1) is well-behaved w.r.t. linear transformations of V , i.e., each $\phi \in \text{GL}(V)$ induces a diffeomorphism $\underline{\phi}$ of $\text{Gr}(V, n)$, and bundle automorphisms of $R(V, n)$, $\text{Gr}(V, n) \times V$, and $N(V, n)$, which cover $\underline{\phi}$.

So, to any vector bundle $\xi : E \rightarrow M$ one can associate a smooth bundle $\text{Gr}(E, n)$ over M , and a short exact sequence of vector bundles over $\text{Gr}(E, n)$,

$$(1.3) \quad \begin{array}{ccccc} R(E, n) & \hookrightarrow & \text{Gr}(E, n) \times_M E & \twoheadrightarrow & N(E, n) \\ & \searrow & \downarrow & \swarrow & \\ & & \text{Gr}(E, n) & & \end{array}$$

such that $\text{Gr}(E, n)_x = \text{Gr}(E_x, n)$ and the restriction of (1.3) to a point $x \in M$ equals (1.1), with $V = E_x$. Furthermore, if $V\text{Gr}(E, n)$ denotes the vertical tangent bundle of $\text{Gr}(E, n)$, then

$$V\text{Gr}(E, n) \cong \text{Hom}_{\text{Gr}(E, n)}(R(E, n), N(E, n)) = R^\vee(E, n) \otimes_{\text{Gr}(E, n)} N(E, n).$$

Finally, for any vertical distribution Δ on $\text{Gr}(E, n)$, i.e., a vector sub-bundle $\Delta \leq V\text{Gr}(E, n)$, a projective sub-bundle

$$(1.4) \quad \text{char}^{\mathbb{R}}(\Delta) \leq \mathbb{P}(R(E, n)^\vee)$$

exists, such that $\text{char}^{\mathbb{R}}(\Delta)|_{\text{Gr}(E_x, n)} = \text{char}^{\mathbb{R}}(\Delta|_{\text{Gr}(E_x, n)})$.

It is natural to call $\text{Gr}(E, n)$ the *Grassmann bundle* (of ξ), and (1.3) its *universal sequence*. Accordingly, $\text{char}^{\mathbb{R}}\Delta$ is the *bundle of characteristics* of the vertical distribution Δ .

Example 2 (Jet bundles). Let $\xi = \tau_M$ be the tangent bundle to M , $J^1(M, n) := \text{Gr}(TM, n)$ and $\tilde{TM} := \pi_{1,0}^\circ(TM)$, and rewrite (1.3) by decorating R and N with an index “1”. The so-obtained diagram

$$\begin{array}{ccccc} R^1(M, n) & \hookrightarrow & \tilde{TM} & \twoheadrightarrow & N^1(M, n) \\ & \searrow R^1 & \downarrow \tilde{\tau}_M & \swarrow N^1 & \\ & & J^1(M, n) & & \\ & & \downarrow \pi_{1,0} & & \\ & & M & & \end{array}$$

contains, besides the definition of the 1st order jet bundle of n -dimensional submanifolds of M , also the main structures necessary to deal with 1st order nonlinear PDEs and to define higher-order jet spaces as well. In the theory of PDEs, the canonical and the normal bundles $R^1(M, n)$ and $N^1(M, n)$ are sometimes called “distributions” (like, e.g., in [1]), and we embrace this terminology. Indeed, $R^1(M, n)$ can be interpreted as a *relative* distribution, in the sense that it is generated by *relative* vector fields (see [15]) w.r.t. $\pi_{1,0}$ (hence, “ R ” stands for “relative”). Also $N^1(M, n)$ can be seen as a distribution, since, for any n -dimensional submanifold $L \subseteq M$, the restricted distribution $N^1(M, n)|_{L^{(1)}}$ is isomorphic to the *normal* (in the usual sense) distribution of L .

In this perspective, the *Cartan distribution* \mathcal{C}^1 plays the role of the “absolute” counterpart of the relative distribution $R^1(M, n)$, i.e.,

$$(1.5) \quad \mathcal{C}_\theta^1 \stackrel{\text{def}}{=} (d_\theta \pi_{1,0})^{-1}(R^1(M, n)_\theta), \quad \theta \in J^1(M, n).$$

Example 3 (Characteristics of a PDE). When \mathcal{E} is a 1st order PDE, i.e., a closed sub-bundle $\mathcal{E} \subseteq J^1(M, n)$, then the vertical distribution $V\mathcal{E}$ over \mathcal{E} can be seen as the restriction to \mathcal{E} of a vertical distribution Δ on $J^1(M, n)$. Then, thanks to (1.4)

above one can speak of the *bundle of characteristics* of \mathcal{E} ,¹ unequivocally defined as $\text{char}^{\mathbb{R}}(\mathcal{E}) := \text{char}^{\mathbb{R}}(\Delta)|_{\mathcal{E}}$.

2. THE POLAR DISTRIBUTION ON ISOTROPIC PLANES

We collect here some necessary algebraic facts about bilinear forms. V and W are finite-dimensional vector spaces and, given a subspace $L \subseteq V$, L^\dagger is the *annihilator* of L in $\text{Hom}(V, W)$, i.e., the linear subspace of $\text{Hom}(V, W)$ given by those homomorphisms which vanish when restricted to L .

Let $\Pi \in V^\vee \wedge V^\vee \otimes W$ be a W -valued skew-symmetric bilinear form on V , take an element $L \in \text{Gr}(V, n)$, and fix an its complement L^c . Then Π determines a “restriction to the graph of h ” map

$$\begin{aligned} \text{Hom}(L, L^c) &\xrightarrow{e_\Pi} L^\vee \wedge L^\vee \otimes W, \\ h &\longmapsto \Pi \circ (\text{id}_L + h), \end{aligned}$$

which is a polynomial of degree 2 on $\text{Hom}(L, L^c)$ with values in $L^\vee \wedge L^\vee \otimes W$ since, $\forall x, y \in L$,

$$(e_\Pi(h))(x, y) = \Pi(x, y) + \Pi(x, h(y)) + \Pi(h(x), y) + \Pi(h(x), h(y)).$$

Hence, the differential $d_0 e_\Pi$ coincides with the linear part of (2), viz.

$$(2.1) \quad d_0 e_\Pi : h \in \text{Hom}(L, L^c) \longmapsto \Pi(\cdot, h(\cdot)) + \Pi(h(\cdot), \cdot) \in L^\vee \wedge L^\vee \otimes W.$$

Now observe that the subset

$$\mathcal{I}_n(\Pi) \stackrel{\text{def}}{=} \{L \in \text{Gr}(V, n) \mid \Pi|_L = 0\},$$

made of (Π) -isotropic n -planes in V , is a quadric variety, since it coincides with $e_\Pi^{-1}(\{0\})$ in the affine neighborhood $\text{Hom}(L, L^c)$. In other words, e_Π can be seen as a smooth map on $\text{Gr}(V, n)$, which is defined only on $\text{Hom}(L, L^c)$, where it becomes polynomial of degree 2, and its differential at L is given by (2.1). Interestingly, if L is isotropic, an analogous formula as (2.1) for $d_L e_\Pi$ can be given, which does not require choosing a complement of L . To this end, for any $L \in \text{Gr}$ denote by $\Pi_L^\#$ the homomorphism

$$\begin{aligned} L &\longrightarrow V^\vee \otimes W, \\ l &\longmapsto \Pi_L^\#(l) := \Pi(l, \cdot), \end{aligned}$$

and notice that $L \in \mathcal{I}_n(\Pi)$ if and only if $\text{im } \Pi_L^\# \subseteq L^\dagger = \left(\frac{V}{L}\right)^\vee \otimes W$. Hence, if L is isotropic, then $d_L e_\Pi$ reads

$$\begin{aligned} T_L \text{Gr}(V, n) = \text{Hom}\left(L, \frac{V}{L}\right) &\xrightarrow{d_L e_\Pi} L^\vee \wedge L^\vee \otimes W, \\ (2.2) \quad h &\longmapsto \left((x, y) \longmapsto \Pi_L^\#(x)(h(y)) - \Pi_L^\#(y)(h(x))\right). \end{aligned}$$

For any $L \in \mathcal{I}_n(\Pi)$ put

$$\begin{aligned} T_L &\stackrel{\text{def}}{=} \ker d_L e_\Pi = \{h \in \text{Hom}_R\left(L, \frac{V}{L}\right) \mid \Pi_L^\#(x)(h(y)) - \Pi_L^\#(y)(h(x)) = 0 \quad \forall x, y \in L\}, \\ D_L &\stackrel{\text{def}}{=} \{h \in \text{Hom}_R\left(L, \frac{V}{L}\right) \mid \Pi_L^\#(\cdot)(h(\cdot)) = 0\}. \end{aligned}$$

Clearly, D_L is a linear subspace of T_L , which in its turn is a linear subspace of $T_L \text{Gr}(V, n)$. Hence, $T : L \longmapsto T_L$ and $D : L \longmapsto D_L$ can be seen as distributions on $\text{Gr}(V, n)$ along the subset $\mathcal{I}_n(\Pi)$. Let $\check{\mathcal{I}}_n(\Pi)$ be the set of smooth points of $\mathcal{I}_n(\Pi)$, i.e., those where (2.2) has maximal rank. Then $T|_{\check{\mathcal{I}}_n(\Pi)}$ is the tangent bundle of $\check{\mathcal{I}}_n(\Pi)$, and $\mathbb{D}^\Pi \stackrel{\text{def}}{=} D|_{\check{\mathcal{I}}_n(\Pi)}$ is a distribution on $\check{\mathcal{I}}_n(\Pi)$.

¹This is just the “characteristic variety” à la Bryant&Griffiths [2], recast in a bundle-theoretic context.

Definition 4. \mathbb{D}^Π is the *polar distribution*² on $\mathcal{I}_n(\Pi)$.

Remark 5 (Geometrical interpretation). Let L be isotropic and h an its deformation, i.e., $h \in \text{Hom}(L, \frac{V}{L})$. This means that h determines a 1-parameter family $\{L_\epsilon\}$ of n -planes, where

$$L_\epsilon := \text{graph}(\epsilon h) = \{x + \epsilon h(x) \mid x \in L\}$$

and $L_0 = L$. Now, condition for L_ϵ to be isotropic is

$$\Pi|_{L_\epsilon} = 0 \Leftrightarrow (\Pi(x, h(y)) + \Pi(h(x), y))\epsilon + \Pi(h(x), h(y))\epsilon^2 = 0 \quad \forall x, y \in L.$$

Hence, $h \in T_L$ if and only if $\{L_\epsilon\}$ is to 1st order made of involutive planes. On the other hand, the family $\{L_\epsilon\}$ determines an enveloping manifold³ whose tangent space at h (seen as a subspace of V) is known as the osculator $\text{osc}(h)$ of h , but condition $h \in T_L$ is not enough to guarantee that $\text{osc}(h)$ is itself isotropic.

However, $h \in D_L$ if and only if $\text{osc}(h)$ has this nice property: for each isotropic subspace $W \subseteq \text{osc}(h)$, $L + W$ is isotropic as well. This clarifies the geometrical content of Definition 4: D_L is made by those deformations of L which are to 1st order orthogonal to L , with respect to Π (i.e., it is the “polar space” in the sense of [2]).

Remark 6 (Duality). Let $\dim V = n$. In view of the isomorphism $\text{Gr}(V, n-1) \cong \mathbb{P}(V^\vee)$, the subset $\mathcal{I}_{n-1}(\Pi) \subseteq \text{Gr}(V, n-1)$ of isotropic hyperplanes in V is in fact a projective subspace of $\mathbb{P}(V^\vee)$, its equation being $\alpha \wedge \Pi = 0$, where $\alpha \in V^\vee$.

In order to link the purely algebraic considerations of this section with the equation of involutive planes which will be discussed below, it is enough to “globalize” the above constructions, by starting with a tensor $\Pi \in \Gamma(\xi^\vee \wedge \xi^\vee \otimes \eta)$, where $\xi : Q \rightarrow M$ and $\eta : Q' \rightarrow M$ are vector bundles. Then $\mathcal{I}_n(\Pi)$ is a sub-bundle of $\text{Gr}(Q, n) \rightarrow Q$ and \mathbb{D}^Π is a vertical distribution on $\tilde{\mathcal{I}}_n(\Pi)$. In our subsequent analysis, $Q = \mathcal{C}^k$ will be the bundle (over $J^k(E, n)$) of Cartan vectors and $\Pi = \Pi^k$, the metasymplectic form on \mathcal{C}^k .

3. THE EQUATION OF INVOLUTIVE PLANES

Let Δ be a distribution on E , and $\Omega \in \Lambda^1(E) \otimes_E P$, with P a $C^\infty(E)$ -module, an its dual P -valued form, i.e., such that $\Delta = \ker \Omega$. Denote by $\Pi \in \Lambda^2(\Delta^\vee) \otimes_E \frac{\Delta^{(1)}}{\Delta}$ the curvature form of Δ , where $\Delta^{(1)}$ is the 1st derived distribution.

Definition 7. A tangent plane R to E is called *involutive* if

$$\Omega|_R = 0, \quad \Pi|_R = 0.$$

The totality of n -dimensional involutive planes of E is the *equation of involutive n -dimensional planes* of E , and denoted by $\mathcal{I}_n(\Delta)$. If E is fibered, then $\tilde{\mathcal{I}}_n(\Delta) \stackrel{\text{def}}{=} \{R \in \mathcal{I}_n(\Delta) \mid R \text{ is horizontal}\}$ is an open and dense subset of $\mathcal{I}_n(\Delta)$ called the equation of *horizontal involutive n -planes*.

A necessary step to reveal that the equation of involutive planes is equipped with a canonical distribution, is to show its equivalence with the Grassmannian of isotropic planes.

Lemma 8. $\mathcal{I}_n(\Delta) = \mathcal{I}_n(\Pi)$ and, in the fibered case, $\tilde{\mathcal{I}}_n(\Delta) \subseteq \tilde{\mathcal{I}}_n(\Pi)$.

Proof. Regard Δ as a bundle over E , and Π as a skew-symmetric bilinear form on it, with values in the bundle Q' whose module of sections is $\frac{\Delta^{(1)}}{\Delta}$ (see the end of previous Section 2). Then $\mathcal{I}_n(\Pi)$ is the sub-bundle of $\text{Gr}(\Delta, n)$ made of n -planes

²The name is mutated by the standard terminology of the theory of quadric surfaces.

³Called *block* by some authors. See, e.g., [5].

on which Π vanishes or, which is the same, the sub-bundle of $\text{Gr}(TE, n) = J^1(E, n)$ made of n -planes on which both Ω and Π vanishes, i.e., precisely the equation $\mathcal{I}_n(\Delta)$.

It remains to be observed that an horizontal involutive n -plane is always surrounded by a neighborhood of such planes. \square

Example 9 (Coordinates). Let $E = \{(x^i, u^j)\}$, and Δ given by means of 1-forms

$$\Delta = \cap_{A \in \mathbb{A}} \ker \omega^A,$$

with $\omega^A = \omega_i^A dx^i + \omega_j^A du^j$. Then

$$(3.1) \quad \mathcal{I}_n(\Delta) = \{\theta \in J^1(E, n) \mid \omega^A|_{R_\theta} = 0, d\omega^A|_{R_\theta} = 0, \quad \forall A \in \mathbb{A}\}$$

is locally given by the system of equations $\{f_i^A = 0, f_{ij}^A = 0 \mid i, j = 1, 2, \dots, n, A \in \mathbb{A}\}$, where⁴

$$(3.2) \quad f_i^A = \omega_i^A + \omega_j^A u_i^j,$$

$$(3.3) \quad f_{ij}^A = \omega_{[i;j]}^A + \omega_{[[i;k]]}^A u_j^k + \omega_{[h;k]}^A u_j^h u_i^k.$$

To prove (3.2) and (3.3) it is enough to recall that a point $\theta = (x^i, u^\alpha, u_i^\alpha)$ correspond to the plane R_θ generated by the n vectors $\partial_{x^i}|_\theta + u_i^\alpha \partial_{u^\alpha}|_\theta$, $i = 1, 2, \dots, n$, and write down conditions (3.1) in terms of the coordinates of θ .

Remark 10. Let $\mathcal{E} \subset J^1(E, n)$ be given just by (3.2) alone. Then $\mathcal{I}_n(\Delta) = \pi_{2,1}(\mathcal{E}^{(1)})$, i.e., (3.3) are differential consequences of (3.2).

Example 11. If \mathcal{C}^k is the Cartan distribution on $J^k(E, n)$, then $\mathcal{I}_n(\mathcal{C}^k)$ is the closure of $J^{k+1}(E, n)$ in $J^1(J^k(E, n), n)$. Adherent points corresponds to the so-called singular R -planes.⁵

Example 12. Let \mathcal{C}^k be as in Example 11. Then $\tilde{\mathcal{I}}_n(\mathcal{C}^k) = J^{k+1}(E, n)$.

Remark 13. Leaves of $\mathcal{I}_r^{(\infty)}(\Delta)$ are in one-to-one correspondence with n -dimensional involutive submanifolds of Δ (see Remark 10). If Δ is integrable, then $\mathcal{I}_r^{(\infty)}(\Delta)$ can be seen as the orbit space of the leaves of Δ .

Lemma 8 above suggests an obvious flag-theoretic generalization of the equation of involutive planes, namely

$$\mathcal{I}_{n,n'}(\Delta) \stackrel{\text{def}}{=} (\mathcal{I}_n(\Delta) \times_E \mathcal{I}_{n'}(\Delta)) \cap \text{Gr}(\Delta, n, n'),$$

where $\text{Gr}(\Delta, n, n')$ is the flag bundle associated with the vector bundle Δ over E .

Definition 14. $\mathcal{I}_{n,n'}(\Delta)$ is the *bundle of flags of involutive planes* of the distribution Δ .

4. APPLICATION TO JET BUNDLES

Definitions and results of Section 3 have interesting applications in the context of the geometrical theory of PDEs. Here involutive planes of dimension $n - s$ are called of *type s*. More precisely, the 1st order PDE

$$(4.1) \quad \mathcal{I}_{n-s}(\mathcal{C}^k) = \{\Theta \in J^1(J^k(E, n), n - s) \mid \Omega^k|_{R_\Theta} = 0, \Pi^k|_{R_\Theta} = 0\}$$

(resp., its open and dense subset $\tilde{\mathcal{I}}_{n-s}(\mathcal{C}^k)$) is called the *equation of s-type involutive subspaces* (resp., *s-type horizontal involutive subspaces*) of \mathcal{C}^k , where Ω^k is the N -valued Cartan form and Π^k its curvature form.

Corollary 15. $\mathcal{I}_{n-s}(\mathcal{C}^k)$ is equipped with the natural, i.e., contact-invariant, polar distribution \mathbb{D}^{Π^k} .

⁴Physicists' notations have been used here: “[i, j]” means sew-symmetrization of indexes, and “ $;$ ” means derivative with respect to the coordinate labeled by i .

⁵Firstly studied by Vinogradov in the context of singular and multivalued solutions [18, 17].

Example 16. $\check{\mathcal{I}}_n(C^k)$ is the $k + 1^{\text{st}}$ order jet space $J^{k+1}(E, n)$, and the restricted projection $\pi_{1,0}|_{\check{\mathcal{I}}_n(C^k)}$ is $\pi_{k+1,k}$. i.e., jets of order $k + 1$ are nothing but involutive subspaces of C^k of type 0. Notice that, in this case, the polar distribution is trivial.

Pass now to the next case, i.e., of type 1 subspaces, and introduce the following commutative diagram

$$(4.2) \quad \begin{array}{ccccc} & & \mathcal{I}_{n,n-1}(\Pi^{k-1}) & & \\ & \swarrow c & \downarrow q & \searrow p & \\ J^k(E, n) & & & & \mathcal{I}_{n-1}(\Pi^{k-1}) \\ & \searrow & \downarrow & \swarrow & \\ & & J^{k-1}(E, n) & & \end{array}$$

Observe that for any $\theta \in J^k(E, n)$, $c_\theta = \text{Gr}(R_\theta, n-1) = \mathbb{P}R_\theta^\vee$, i.e., $\mathcal{I}_{n,n-1}(\Pi^{k-1})$ is the total space of the bundle $\mathbb{P}R^\vee$ over $J^k(E, n)$. This provides a transparent geometrical interpretation of flags of involutive subspaces, and shows that $\mathcal{I}_{n,n-1}$ is the natural environment for characteristics (see Corollary 17 below).

Corollary 17. *Let \mathcal{E} be a k^{th} order PDE. Then $\text{char}^{\mathbb{R}}(\mathcal{E})$ is a sub-bundle of $c|_{\mathcal{E}}$.*

Diagram (4.2) shows the relationship between characteristics and so-called *singular solutions*, since involutive subspaces of type s can be seen as the “shadows” of singular R -planes of type s . Details are as follows.

Definition 18. Elements of $\mathcal{I}_n(C^k) \setminus \check{\mathcal{I}}_n(C^k)$ are called *singular R -planes*.⁶ An n -dimensional, C^k -integral submanifold W of \mathcal{E} , is called a *singular solution* (at $\theta \in \mathcal{E}$), iff $T_\theta W$ is a singular R -plane. The *type* of a singular R -plane $R \leq T_\theta J^k$ is the codimension in R_θ of $d_\theta \pi_{k,k-1} R$. The *type of the singularity* of W (at θ) is the type of $T_\theta W$.

Remark 19. The projection $\pi_{k+1,k}$ can be regarded as the tangent map $T\pi_{k,k-1}$, sending horizontal R -planes on J^k into horizontal R -planes on J^{k-1} , i.e., acting between $\check{\mathcal{I}}_n(C^k)$ and $\check{\mathcal{I}}_n(C^{k-1})$. On the other hand, $T\pi_{k,k-1}$ cannot act between $\mathcal{I}_n(C^k)$ and $\mathcal{I}_n(C^{k-1})$, since the projection of a singular R -plane is still involutive, but fails to be n -dimensional.

However, if $\mathcal{I}_n(C^k)$ is seen as a stratified manifold, where the s^{th} stratum is given by singular R -planes of type s , with $s = 0, 1, \dots, n$, then $T\pi_{k,k-1}$ defines a projection of the s -stratum of $\mathcal{I}_n(C^k)$ onto $\check{\mathcal{I}}_{n-s}(C^{k-1})$ (indeed, the projection of an involutive space is always horizontal).

In other words, $\check{\mathcal{I}}_{n-s}(C^{k-1})$, plays the role of “empty equation”, i.e., the “receptacle” for all projections of *all* singular solutions of type s of *all* k^{th} order PDEs. The next logical step is to confine oneself to a concrete k^{th} order PDE \mathcal{E} , and consider the elements of $\check{\mathcal{I}}_{n-s}(C^{k-1})$ which are the projection of a singular solution of \mathcal{E} of type s . More precisely, denoting by \mathcal{E}_{k-1} the projection of \mathcal{E} onto J^{k-1} , the projection of the s -stratum of $\mathcal{I}_n(C^k_{\mathcal{E}})$ onto $\check{\mathcal{I}}_{n-s}(C^{k-1}_{\mathcal{E}_{k-1}})$,

$$\Sigma_{[s]}\mathcal{E} \stackrel{\text{def}}{=} \{T\pi_{k,k-1}(W) \mid W \in \mathcal{I}_n(C^k_{\mathcal{E}}) \text{ is of type } s\} \subset \check{\mathcal{I}}_{n-s}(C^{k-1}_{\mathcal{E}_{k-1}})$$

is a 1^{st} order PDE on \mathcal{E}_{k-1} called the s -type *singularity equation* of \mathcal{E} .

Remark 20. The projection of a regular solution of \mathcal{E} on \mathcal{E}_{k-1} , is a solution of \mathcal{E}_{k-1} , while the projection of a singular solution of \mathcal{E} (of type s at θ) is a singular submanifold of \mathcal{E}_{k-1} , and the tangent space to its singularity locus at $\pi_{k,k-1}(\theta)$ is

⁶This definition is due to Vinogradov [18].

an $(n - s)$ -dimensional involutive subspace of $C_{\mathcal{E}^{k-1}}^{k-1}$. The totality of such tangent spaces is $\Sigma_{[s]}\mathcal{E}$.

The link between singularity equation and characteristics easily follows from Diagram (4.2) above.

Corollary 21. $\Sigma_{[1]}\mathcal{E}$ is the projection of $\text{char}_{\mathcal{E}}^{\mathbb{R}}$ via the map p of Diagram (4.2). The next result is a source of contact invariant for nonlinear PDEs.

Corollary 22. Singularity equation $\Sigma_{[s]}\mathcal{E}$ is equipped with a natural, i.e., contact invariant, polar distribution, obtained by restriction of the polar distribution $\mathbb{D}^{\Pi^{k-1}}$ on $\mathcal{I}_{n-s}(C^{k-1})$.

5. COMPUTATION OF \mathbb{D}^{Π} ON SOME EQUATIONS OF INVOLUTIVE LINES

The equation of involutive lines is particularly simple, since all lines are involutive, i.e., $\mathcal{I}_1(\Delta) = \mathbb{P}(\Delta)$ for *any* distribution Δ . Hence, in this case, \mathbb{D}^{Π} is a distribution on the projectivized bundle $\mathbb{P}(\Delta)$, where Π is the curvature of Δ .

While working on the present subject, the authors developed the opinion that, if Π is the metasymplectic structure on a “empty equation”, i.e., a jet space, then the corresponding polar distribution \mathbb{D}^{Π} on it should take a particularly simple form. In this context, “particularly simple” means isomorphic to the Cartan distribution on a PDE of the simplest form, i.e., again an empty one, expectedly with a smaller number of independent variables. This guess can be formalized as a simple conjecture,⁷ which can be proved true in few cases, collected in this last section.

Conjecture 1. Let \mathcal{C} be the Cartan distribution on $J^k(E, n)$, and Π its metasymplectic structure. Then an $N > n$ exists such that, for any $\theta \in J^k(E, n)$, the projective space $\mathbb{P}(\mathcal{C}_{\theta})$ is (locally) isomorphic to $J^k(\mathbb{R}^N, n - 1)$, and the polar distribution \mathbb{D}^{Π} coincides with the Cartan distribution on $J^k(\mathbb{R}^N, n - 1)$.

In Subsection 5.1 we show that the polar distribution on a 1st order *empty* PDE in n independent variables identifies with the Cartan distribution on an *empty* 1st order PDE in $n - 1$ independent variables.

In Subsection 5.2 we pass to a *nonempty* 1st order PDE \mathcal{E} , and show how to compute the family of polynomials determining the corresponding polar distribution on the equation of the involutive planes of \mathcal{E} , from the smooth function(s) cutting out \mathcal{E} . This is a nice consequence of Conjecture 1, which can be stated in full generality.

Corollary 23. Let $\mathcal{E} \subseteq J^k(E, n)$ be a PDE, $\mathcal{C}^{\mathcal{E}}$ its Cartan distribution, and $\Pi_{\mathcal{E}}$ the corresponding metasymplectic structure. Then, for any $\theta \in \mathcal{E}$, the projective space $\mathbb{P}(\mathcal{C}_{\theta}^{\mathcal{E}})$, equipped with the polar distribution $\mathbb{D}^{\Pi_{\mathcal{E}}}$, is a PDE in $J^k(\mathbb{R}^N, n - 1)$. Moreover, the so-obtained family $\{\mathbb{P}(\mathcal{C}_{\theta}^{\mathcal{E}})\}_{\theta \in \mathcal{E}}$ of PDEs is a contact invariant of \mathcal{E} , in the sense that any contact transformation $F : \mathcal{E} \rightarrow \mathcal{E}$ induces an isomorphism between the PDEs $\mathbb{P}(\mathcal{C}_{\theta}^{\mathcal{E}})$ and $\mathbb{P}(\mathcal{C}_{F(\theta)}^{\mathcal{E}})$. Subsection 5.3 outlines a strategy to obtain a complete proof of Conjecture 1: to regard $k + 1$ st order jets as 1st order PDEs and then use the result of Subsection 5.1 as the induction basis and that of Subsection 5.2 as the induction step. Due to the complexity of the expression involved (see (5.14) and (5.15)), the authors did not complete the task, but the subsequent examples show it should be conceptually simple. Example 25 proves Conjecture 1 for a scalar 2nd order *empty* PDE in two independent variables, highlighting the role of a good induction step for the general proof. The result is in fact valid for *all* scalar equations in two independent variables (Example 27), but its proof is

⁷This conjecture was brought to the attention of the first author by his Ph.D. advisor, prof. A. M. Vinogradov.

technical, and gives no hints for the general case (for this reason, the proof of its key Lemma 28, which is rather lengthy and can be found in the second author's Ph.D. thesis, is omitted). In the Example 29 (which is an application of Corollary 23), results of Example 27 are used to describe the polar distribution on the singularity equations of a third order nonlinear PDE in two independent variables, showing the importance of Corollary 21 in these kinds of computations.

5.1. Computation of \mathbb{D}^Π for \mathcal{I}_1 on 1st order jets. Let E be a manifold of dimension $n + m$. Denote by \mathcal{C} and Π the Cartan distribution on $J^1(E, n)$ and the metasymplectic structure on \mathcal{C} , respectively. Choose local coordinates (x^i, u^j, u_i^j) , with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Then, locally,

$$(5.1) \quad \mathcal{C} = \text{Span} \left\{ D_i, \partial_{u_i^j} \right\}$$

and

$$\Pi = dx^i \wedge du_i^j \otimes \partial_{u^j}.$$

Since all lines lying in \mathcal{C} are Π -involutive, $I_1(\Pi^1) = \mathbb{P}\mathcal{C}$. Hence, \mathbb{D}^Π is a vertical distribution on the bundle $\mathbb{P}\mathcal{C} \rightarrow J^1(E, n)$. Since $J^1(E, n)$ is homogeneous w.r.t. contact transformations, it is enough to describe the restriction of \mathbb{D}^Π to a generic fiber $\mathbb{P}\mathcal{C}_\theta$, with $\theta \in J^1(E, n)$. Let $L = \text{Span} \{l\} \in \mathbb{P}\mathcal{C}_\theta$, with

$$(5.2) \quad l = D_1|_\theta + b^\alpha D_\alpha|_\theta + f_i^j \partial_{u_i^j}|_\theta$$

where $\alpha \in \{2, \dots, n\}$. Then

$$(5.3) \quad (b^2, \dots, b^n, f_1^j, \dots, f_n^j)$$

is a coordinate system for the open neighborhood $\text{Hom}_{\mathbb{R}}(L, L^c)$ of L , in the projective space $\mathbb{P}\mathcal{C}_\theta = \mathbb{R}\mathbb{P}^{n(m+1)-1}$, where

$$L^c := \text{Span} \left\{ D_\alpha|_\theta, \partial_{u_i^j}|_\theta \right\}$$

is a complement of L . Since $\text{Hom}_{\mathbb{R}}(L, L^c)$ identifies with $T_L(\mathbb{P}\mathcal{C}_\theta)$ by means of the correspondence

$$(h : l \mapsto B^\alpha D_\alpha|_\theta + F_i^j \partial_{u_i^j}|_\theta) \leftrightarrow \xi_h = B^\alpha \partial_{b^\alpha}|_\theta + F_i^j \partial_{f_i^j}|_\theta,$$

the fact that ξ_h belongs to \mathbb{D}_L^Π reflects on a condition on the coefficients B^α and F_i^j . More precisely,

$$\begin{aligned} \xi_h \in \mathbb{D}_L^\Pi &\Leftrightarrow \Pi_\theta(l, h(l)) = 0 \\ &\Leftrightarrow \Pi_\theta(D_1|_\theta + b^\alpha D_\alpha|_\theta + f_i^j \partial_{u_i^j}|_\theta, B^\alpha D_\alpha|_\theta + F_i^j \partial_{u_i^j}|_\theta) = 0 \\ &\Leftrightarrow F_1^j \Pi \left(D_1, \partial_{u_i^j} \right)|_\theta + b^\alpha F_\alpha^j \Pi \left(D_\alpha, \partial_{u_i^j} \right)|_\theta + f_\alpha^j B^\alpha \Pi \left(\partial_{u_i^j}, D_\alpha \right)|_\theta = 0 \\ &\Leftrightarrow (F_1^j + b^\alpha F_\alpha^j - f_\alpha^j B^\alpha) \partial_{u^j}|_\theta = 0 \\ (5.4) \quad &\Leftrightarrow F_1^j = f_\alpha^j B^\alpha - b^\alpha F_\alpha^j \quad \forall j = 1, 2, \dots, m. \end{aligned}$$

Equations (5.4) shows that the elements of \mathbb{D}_L^Π are in one-to-one correspondence with the $(n-1)m$ -tuples $(B^2, \dots, B^n, F_2^j, \dots, F_n^j)$. In particular, we obtain a basis

$$\xi_2, \dots, \xi_n, \xi_2^j, \dots, \xi_n^j$$

of \mathbb{D}_L^Π by choosing $F_\alpha^j = 0$ and $B^\alpha = \delta_\beta^\alpha$ for the X_β 's (with $\beta = 2, 3, \dots, n$), and then choosing $B_\alpha = 0$ and $F_\alpha^j = \delta_\alpha^\beta \delta_k^j$ for the X_k^β 's (with $\beta = 2, 3, \dots, n$ and $k = 1, 2, \dots, m$). Notice that the values at L of the vector fields

$$(5.5) \quad X_\alpha := \partial_{b^\alpha} + f_\alpha^j \partial_{f_1^j},$$

$$(5.6) \quad X_j^\alpha := \partial_{f_\alpha^j} - b^\alpha \partial_{f_1^j},$$

are, by construction, ξ_α and ξ_j^α , respectively, i.e., \mathbb{D}^Π is spanned by (5.5) and (5.6). Moreover,

$$(5.7) \quad [X_\alpha, X_j^\beta] = -2\delta_\alpha^\beta \partial_{f_1^j}.$$

Observe that (5.5) resembles a total derivative, (5.6) is almost a vertical vector field, and (5.7) is reminiscent of the standard commutation relation of the standard non-holonomic frame of a first order jet space. The idea is to look for a coordinate system which “turns right” formulas (5.5), (5.6) and (5.7). To this end, let

$$(5.8) \quad (y^2, \dots, y^n, v^1, \dots, v^m, v_2^j, \dots, v_n^j)$$

be a new coordinates on $\mathbb{P}\mathcal{C}$, given by

$$(5.9) \quad b^\alpha := y^\alpha,$$

$$(5.10) \quad f_\alpha^j := v_\alpha^j,$$

$$(5.11) \quad f_1^j := 2v^j - v_\alpha^j y^\alpha.$$

Then it is immediate to see that

$$\begin{aligned} \partial_{y^\alpha} + v_\alpha^j \partial_{v^j} &= X_\alpha, \\ \partial_{v_\alpha^j} &= X_j^\alpha, \\ -\delta_\alpha^\beta \partial_{v^j} &= [X_\alpha, X_j^\beta]. \end{aligned}$$

In other words, (5.9), (5.10) and (5.11) define a local diffeomorphism between $\mathbb{P}\mathcal{C}_\theta$ and $J^1(\mathbb{R}^{n-1+m}, n-1)$, which sends \mathbb{D}^Π into the Cartan distribution of $J^1(\mathbb{R}^{n-1+m}, n-1)$.

5.2. Computation of \mathbb{D}^Π for \mathcal{I}_1 on 1st order PDEs. In the same setting of Subsection 5.1, the dual basis of (5.1) is

$$(5.12) \quad \mathcal{C}^\vee = \text{Span} \left\{ dx^i, du_i^j \right\}.$$

Let now $\mathcal{E} \subseteq J^1(E, n)$ be the PDE determined by $\phi \in C^\infty(J^1(E, n))$, and put $\mathcal{C}^\mathcal{E} := \mathcal{C}|_\mathcal{E}$. Accordingly, $\Pi^\mathcal{E}$ will denote the metasymplectic structure on $\mathcal{C}^\mathcal{E}$. In terms of coordinates (5.12), $\alpha_\phi := d_\theta \phi|_{\mathcal{C}_\theta} \in \mathcal{C}_\theta^\vee$ reads

$$\alpha_\phi = D_i(\phi)(\theta) d_\theta x^i + \frac{\partial \phi}{\partial u_i^j}(\theta) d_\theta u_i^j.$$

Hence, a line $l \in \mathbb{P}(\mathcal{C}_\theta)$ belongs to $\mathcal{I}_1(\mathcal{C}_\theta^\mathcal{E}) = \mathbb{P}(\ker \alpha_\phi)$ if and only if $\alpha_\phi(l) = 0$. More precisely, if l is given by (5.2), then

$$(5.13) \quad \alpha_\phi(l) = D_1(\phi)(\theta) + b^\alpha D_\alpha(\phi)(\theta) + f_i^j \frac{\partial \phi}{\partial u_i^j}(\theta),$$

and the right-hand side of (5.13) is a polynomial in the coordinates (5.3) which determines $\mathcal{I}_1(\mathcal{C}_\theta^\mathcal{E})$. Passing to the coordinates (5.8), the main result of Subsection 5.1 guarantees that $\mathcal{I}_1(\mathcal{C}_\theta^\mathcal{E})$ is isomorphic to the equation $\{P_\theta^\phi = 0\}$ in $J^1(\mathbb{R}^{n-1+m}, n-1)$, with

$$P_\theta^\phi(y^\alpha, v^j, v_\alpha^j) = D_1(\phi)|_\theta + D_\alpha(\phi)|_\theta y^\alpha + 2 \frac{\partial \phi}{\partial u_1^j} \Big|_\theta (v^j - y^\alpha v_\alpha^j) + \frac{\partial \phi}{\partial u_\alpha^j} \Big|_\theta v_\alpha^j.$$

In particular, $\mathbb{D}^{\Pi^\mathcal{E}}$ is isomorphic to the restriction of the Cartan distribution on $J^1(\mathbb{R}^{n-1+m}, n-1)$.

5.3. Computation of \mathbb{D}^Π for \mathcal{I}_1 on higher-order jets. $k + 1^{\text{st}}$ order jets can be treated as differential equations on k^{th} order jets. So, results of Subsection 5.2 can be tested in this context.

Example 24 ($k = 1$). Relabel coordinates of $E' = J^1(E, n)$ as follows

$$E' = \{x^i, \mathbf{u}^J\} \quad \text{with } J \in \{1, \dots, m\} \cup (\{1, \dots, n\} \times \{1, \dots, m\}),$$

where

$$\begin{aligned} \mathbf{u}^j &:= u^j, \\ \mathbf{u}^{i,j} &:= u_i^j. \end{aligned}$$

Hence (see Example 12 before), $J^2(E, n) = \{F_l^j = 0, G_l^{i,j} = 0\} \subseteq J^1(E', n)$, where

$$\begin{aligned} F_l^j &:= \mathbf{u}_l^j - \mathbf{u}^{l,j}, \\ G_l^{i,j} &:= \mathbf{u}_l^{i,j} - \mathbf{u}_i^{l,j}. \end{aligned}$$

Observe that the Cartan distribution \mathcal{C} of $J^2(E, n)$ is the restriction to $J^2(E, n)$ of that on $J^1(E', n)$, and denote by Π^2 the corresponding metasymplectic structure. Then (see Subsection 5.2), the distribution \mathbb{D}^{Π^2} on $\mathbb{P}(\mathcal{C}_\theta)$ is isomorphic to the Cartan distribution on $J^1(\mathbb{R}^{n-1+m+nm}, n-1)$, restricted to the system of equations given by the $\frac{mn(n+1)}{2}$ polynomials (the (5.14)'s are symmetric in i, l)

$$(5.14) \quad P_\theta^{F_l^j} := -\mathbf{u}_1^{l,j}(\theta) - \mathbf{u}_\alpha^{l,j}(\theta)y^\alpha + \delta_l^1 \delta_j^j (2\mathbf{v}^J - y^\alpha \mathbf{v}_\alpha^J) + \delta_l^\alpha \delta_j^j \mathbf{v}_\alpha^J,$$

$$(5.15) \quad P_\theta^{G_l^{i,j}} := 2(\delta_j^{(i,j)} \delta_1^l - \delta_j^{(l,j)} \delta_1^i)(\mathbf{v}^J - y^\alpha \mathbf{v}_\alpha^J) + \delta_j^{(i,j)} \delta_\alpha^l \mathbf{v}_\alpha^J,$$

in the $mn(n+1) + n - 1$ variables

$$y^2, \dots, y^n, \dots, \mathbf{v}^J, \dots, \mathbf{v}_2^J, \dots, \mathbf{v}_n^J.$$

Example 25 ($k = 1, n = 2, m = 1$). If $\{x, y, u, p, q\}$ are local coordinates on $J^1(\mathbb{R}^3, 2)$, then the eleven-dimensional jet space $J^1(J^1(\mathbb{R}^3, 2), 2)$ has coordinates

$$x, y, u, p, q, u_x, u_y, p_x, p_y, q_x, q_y,$$

and its eight-dimensional Cartan distribution \mathcal{C} is spanned by

$$D_x, D_y, \partial_{u_x}, \partial_{u_y}, \partial_{p_x}, \partial_{p_y}, \partial_{q_x}, \partial_{q_y}.$$

Hence, the eight-dimensional 2^{nd} order jet space $J^2(\mathbb{R}^3, 2)$ identifies with the equation

$$(5.16) \quad \mathcal{E} = \{F_1 = F_2 = G = 0\}, \quad F_1 := u_x - p, F_2 := u_y - q, G := p_y - q_x.$$

Let

$$(5.17) \quad b, f_x^1, f_y^1, f_x^2, f_y^2, f_x^3, f_y^3$$

be the projective coordinates of the line

$$L := \text{Span} \{ D_x|_\theta + b D_y|_\theta + f_x^1 \partial_{u_x}|_\theta + f_y^1 \partial_{u_y}|_\theta + f_x^2 \partial_{p_x}|_\theta + f_y^2 \partial_{p_y}|_\theta + f_x^3 \partial_{q_x}|_\theta + f_y^3 \partial_{q_y}|_\theta \}$$

in the seven-dimensional projective space $\mathbb{P}(\mathcal{C}_\theta)$. It is convenient to present the four-dimensional distribution \mathbb{D}^Π by means the three 1-forms

$$(5.18) \quad \omega^j := df^j - f_y^j db + b df_y^j.$$

Indeed (see (5.5) and (5.6)), \mathbb{D}^Π is spanned by the four vector fields

$$\partial_b + f_y^j \partial_{f_x^j}, \quad \partial_{f_y^j} - b \partial_{f_x^j},$$

which are also a basis for the common kernel of the forms (5.18). In the present setting, polynomials (5.14) and (5.15) on $\mathbb{P}(\mathcal{C}_\theta)$ read

$$(5.19) \quad P_\theta^{F_1} = f_x^1 - p_x(\theta) - bp_y(\theta),$$

$$(5.20) \quad P_\theta^{F_2} = f_y^1 - q_x(\theta) - bq_y(\theta),$$

$$(5.21) \quad P_\theta^G = f_y^2 - f_x^3.$$

The zero locus of (5.19), (5.20) and (5.21) is precisely $\mathbb{P}(\mathcal{C}_\theta^\mathcal{E})$ (as in Subsection 5.2), and (see Subsection 5.1) $\mathbb{P}(\mathcal{C}_\theta)$ is isomorphic to $J^1(\mathbb{R}^4, 1)$. Hence, (5.19), (5.20) and (5.21) together define the 1st order equation

$$(5.22) \quad v^1 = \frac{t^2}{2}q_y(\theta) + tQ(\theta) + \frac{1}{2}p_x(\theta),$$

$$(5.23) \quad v_t^1 = Q(\theta) + tq_y(\theta),$$

$$(5.24) \quad v_t^2 = 2v^3 - v_t^3t,$$

where $Q := p_y = q_x$.

We claim that $\mathbb{P}(\mathcal{C}_\theta^\mathcal{E})$ is isomorphic to $J^2(\mathbb{R}^2, 1)$. To this end, observe that condition (5.22) is algebraic, i.e., it does not involve derivatives, and that (5.23) is an its differential consequence. This means that $\mathbb{P}(\mathcal{C}_\theta^\mathcal{E})$ can be understood as a codimension-one submanifold sitting in the five-dimensional graph M of the map

$$(t, v^2, v^3, v_t^2, v_t^3) \mapsto \left(t, \frac{t^2}{2}q_y(\theta) + tQ(\theta) + \frac{1}{2}p_x(\theta), v^2, v^3, Q(\theta) + tq_y(\theta), v_t^2, v_t^3 \right).$$

Let $\eta^j = dv^j - v_t^j dt$ the three 1-forms which define the Cartan distribution \mathcal{C}' on $J^1(\mathbb{R}^4, 1)$. Then, by construction, $\eta^1|_M = 0$, and, in terms of the coordinates $(t, v^2, v^3, v_t^2, v_t^3)$ on M , $\mathcal{C}'|_M$ is given by the two 1-forms

$$(5.25) \quad \eta^2 = dv^2 - v_t^2 dt, \quad \eta^3 = dv^3 - v_t^3 dt.$$

In other words, after the change of coordinates

$$\begin{aligned} t &= t, \\ u &= v^2, \\ p &= v^3, \\ u_t &= v_t^2, \\ p_t &= v_t^3, \end{aligned}$$

M becomes isomorphic to the five-dimensional iterated jet spaces $J^1(J^1(\mathbb{R}^2, 1), 1) = \{(t, u, p, u_t, p_t)\}$, and $\mathcal{C}'|_M$ identifies with its Cartan distribution. Since $J^2(\mathbb{R}^2, 1)$ is singled out by the equation $u_t = p$, which, in the new coordinates, reads

$$(5.26) \quad v_t^2 = v^3,$$

it remains to be proved that (5.24) and (5.26) gives the same equation. To this end, observe that (5.24) allows to express v^3 as a function of (t, v^2, v_t^2, v_t^3) or, equivalently, p as a function of (t, u, u_t, p_t) , which can then be taken as internal coordinates on $\mathbb{P}(\mathcal{C}_\theta^\mathcal{E})$. Accordingly, after restriction to $\mathbb{P}(\mathcal{C}_\theta^\mathcal{E})$, forms (5.25) become

$$(5.27) \quad \eta_1 = du - u_t dt,$$

$$(5.28) \quad \eta_2 = \frac{1}{2}d(u_t + p_t t) - p_t dt = \frac{1}{2}(du_t - p_t dt + t dp_t).$$

The fact that the distribution determined by (5.27) and (5.28) is the same as the Cartan distribution on $J^2(\mathbb{R}^2, 1)$ is a consequence of the general Lemma 26 below.

Lemma 26. *Let Δ be the distribution on $\mathbb{R}^4 = \{(t, u, p, q)\}$ given by the two forms*

$$\begin{aligned}\eta_1 &= du - p dt, \\ \eta_2 &= dp - q dt - \alpha t dq,\end{aligned}$$

with $\alpha \in \mathbb{R} \setminus \{1, \frac{1}{2}\}$. Then Δ is isomorphic to the Cartan distribution on $J^2(\mathbb{R}^2, 1)$.

Proof. With the new coordinates

$$\begin{aligned}t &= \bar{t}, \\ u &= (1 - 2\alpha)\bar{u} + \alpha\bar{p}\bar{t}, \\ p &= \alpha\bar{t}\bar{q} + (1 - \alpha)\bar{p}, \\ q &= \bar{q},\end{aligned}$$

one obtains

$$\begin{aligned}\eta_1 &= (1 - 2\alpha)d\bar{u} + \alpha d(\bar{p}\bar{t}) - (\alpha\bar{t}\bar{q} + (1 - \alpha)\bar{p})d\bar{t} \\ &= (1 - 2\alpha)(d\bar{u} - \bar{p}d\bar{t}) + \alpha t(d\bar{p} - \bar{q}d\bar{t})\end{aligned}$$

and

$$\begin{aligned}\eta_2 &= d(\alpha\bar{t}\bar{q} + (1 - \alpha)\bar{p}) - \bar{q}d\bar{t} - \alpha\bar{t}d\bar{q} \\ &= (1 - \alpha)(d\bar{p} - \bar{q}d\bar{t}).\end{aligned}$$

□

Example 27 ($k \geq 1, n = 2, m = 1$). Let $(x, y, u_{a,b} \mid a + b \leq k)$ be local coordinates on $J^k(\mathbb{R}^3, 2)$, and \mathcal{C} its Cartan distribution. Much as (5.3) and (5.17), the projective coordinates

$$(5.29) \quad b, f_0, f_1, \dots, f_k$$

on the $(k + 2)$ -dimensional projective space $\mathbb{P}(\mathcal{C}_\theta)$ identify the line generated by

$$l := D_x|_\theta + b D_y|_\theta + \sum_{l=0}^k f_l \partial_{u_{l,k-l}}|_\theta.$$

The metasymplectic form on \mathcal{C} reads

$$\Pi = \sum_{l=0}^{k-1} (dx \wedge du_{l+1} \otimes \partial_{u_{l,k-l}} + dy \wedge du_{k-l,l+1} \otimes \partial_{u_{k-l,l}})$$

and (see (5.18)) the distribution \mathbb{D}^Π is the common kernel of the k one-forms

$$(5.30) \quad \omega_l \stackrel{\text{def}}{=} df_{l+1} - f_l db + b df_l, \quad l = 0, \dots, k-1.$$

By Lemma 28 below, after a suitable change of coordinates on $\mathbb{P}(\mathcal{C}_\theta)$, the distribution \mathbb{D}^Π coincides with the Cartan distribution on $J^k(\mathbb{R}^2, 1) = \{(y, v, v^1, \dots, v^k)\}$.

Lemma 28. *Change of coordinates*

$$\begin{aligned}b &= y \\ f_l &= l! \sum_{j=0}^l \frac{(j+1)(-y)^{l-j}}{(l-j)!} v_j \quad l = 0, \dots, k\end{aligned}$$

makes the system (5.30) equivalent to

$$dv^j - v^{j+1} dy.$$

Example 29. The equation $\mathcal{E} = \{\phi = 0\}$, with

$$\phi := u_{yyy} - u_{xxy}^2 + u_{xxx}u_{xyy},$$

is a sub-bundle of

$$J^3 := J^3(\mathbb{R}^3, 2) \longrightarrow J^2(\mathbb{R}^3, 2) =: J^2.$$

Indeed, ϕ does not depend on the coordinates of J^2 and as such, for any $\theta \in J^2$, it can be regarded as a function on the fiber J_θ^3 which singles out \mathcal{E}_θ , viz.

$$(5.31) \quad \mathcal{E}_\theta = \{\phi = 0\} \subseteq J_\theta^3.$$

Notice that \mathcal{E}_θ is a quadric hypersurface in the four-dimensional space J_θ^3 . In the sequel, we denote by $\underline{\theta}$ the projection of θ on M , and by $\tilde{\theta} \in J^3$ any point of J_θ^3 .

Let now $\Omega := \text{Span}\{\omega\} \in \mathbb{P}(R_\theta^\vee)$ be the pencil determined by a covector $\omega \in R_\theta^\vee$. By definition,⁸ Ω is a characteristic of \mathcal{E} at $\tilde{\theta}$ if and only if

$$(5.32) \quad \exists e \in N_{\tilde{\theta}} \text{ s.t. } \omega \otimes e \in T_{\tilde{\theta}}\mathcal{E}_\theta.$$

It is well-known⁹ that condition (5.32) above is equivalent to the fact that the third symmetric power ω^3 , which belongs to $S^3(T_{\tilde{\theta}}^\vee M) = TJ_\theta^3$, actually sits in $T_{\tilde{\theta}}\mathcal{E}_\theta$. In other words,

$$\text{char}_{\tilde{\theta}}^{\mathbb{R}}(\mathcal{E}) = \{\Omega \in \mathbb{P}(R_\theta^\vee) \mid \omega^3 \in T_{\tilde{\theta}}\mathcal{E}_\theta\}.$$

Notice that $\mathbb{P}(R_\theta^\vee)$ is one-dimensional, so that correspondence

$$(5.33) \quad a \leftrightarrow \Omega_a := \text{Span}\{-ad_{\tilde{\theta}}x + d_{\tilde{\theta}}y\}$$

is enough to define a coordinate system on $\mathbb{P}(R_\theta^\vee)$. Accordingly, to each value of a it corresponds a vertical tangent vector ξ_a , namely the one associated with $(-ad_{\tilde{\theta}}x + d_{\tilde{\theta}}y)^3$ via the identification $S^3(T_{\tilde{\theta}}^\vee M) = T_{\tilde{\theta}}J_\theta^3$,

$$\xi_a := -a^3 \partial_{u_{xxx}}|_{\tilde{\theta}} + 3a^2 \partial_{u_{xxy}}|_{\tilde{\theta}} - 3a \partial_{u_{xyy}} + \partial_{u_{yyy}}|_{\tilde{\theta}}.$$

Since ξ_a belongs to $T_{\tilde{\theta}}\mathcal{E}_\theta$ if and only if $\xi_a(\phi) = 0$, the subset $\text{char}_{\tilde{\theta}}^{\mathbb{R}}(\mathcal{E}) \subseteq \mathbb{P}(R_\theta^\vee)$ can be (locally) identified with the set of the zeros of the polynomial

$$p_{\tilde{\theta}}(a) := u_{xyy}(\tilde{\theta})a^3 - 6u_{xxy}(\tilde{\theta})a^2 + 3u_{xxx}(\tilde{\theta})a - 1.$$

Passing to the the fold-type singularity equation $\Sigma_{[1]}\mathcal{E} \subseteq J^1(J^2, 1)$, Corollary 21 shows that

$$(5.34) \quad (\Sigma_{[1]}\mathcal{E})_\theta = \{\ker \Omega_a \mid p_{\tilde{\theta}}(a) = 0 \text{ for some } \tilde{\theta} \in \mathcal{E}_\theta\} \subseteq \mathbb{P}(\mathcal{C}_\theta), \quad \forall \theta \in J^2,$$

where by the kernel of a pencil of covectors we mean that of any its representative. Observe that $\ker \Omega_a$ is an element of the projective line $\mathbb{P}(R_{\tilde{\theta}})$, canonically identified with a subset of the four-dimensional projective space $\mathbb{P}(\mathcal{C}_\theta)$. In other words, $(\Sigma_{[1]}\mathcal{E})_\theta$ is the union of all the “dual lines” to the characteristics of \mathcal{E} at all points projecting over θ , i.e., (5.34) can be rewritten as

$$(5.35) \quad (\Sigma_{[1]}\mathcal{E})_\theta = \bigcup_{\tilde{\theta} \in \mathcal{E}_\theta} \bigcup_{a \in p_{\tilde{\theta}}^{-1}(0)} \ker \Omega_a \subseteq \bigcup_{\tilde{\theta} \in \mathcal{E}_\theta} \mathbb{P}(R_{\tilde{\theta}}) \subseteq \mathbb{P}(\mathcal{C}_\theta), \quad \forall \theta \in J^2,$$

which seems to suggest that, in order to compute the singularity equation at θ one has to find all the roots of the polynomials $p_{\tilde{\theta}}$, with $\tilde{\theta} \in \mathcal{E}_\theta$. In fact, in the present example, this step can be circumvented.

⁸See (1) and Example 3.

⁹See, for instance, [2].

In view of the definition (5.33) of Ω_a , its “dual line” is the element of $\mathbb{P}(R_{\tilde{\theta}})$ represented by $\eta_a := D_x|_{\tilde{\theta}} + aD_y|_{\tilde{\theta}}$ and (5.35) reads

$$(5.36) \quad (\Sigma_{[1]}\mathcal{E})_{\theta} = \bigcup_{\tilde{\theta} \in \mathcal{E}_{\theta}} \bigcup_{a \in p_{\tilde{\theta}}^{-1}(0)} \text{Span}\{\eta_a\}, \quad \forall \theta \in J^2.$$

Equipping $\mathbb{P}(\mathcal{C}_{\theta})$ with the same projective coordinates (b, f_0, f_1, f_2) as Example 27 above, $\text{Span}\{\eta_a\}$ is given by

$$\begin{aligned} b &= a, \\ f_0 &= u_{xyy}(\tilde{\theta}) + bu_{yyy}(\tilde{\theta}), \\ f_1 &= u_{xxy}(\tilde{\theta}) + bu_{xyy}(\tilde{\theta}), \\ f_2 &= u_{xxx}(\tilde{\theta}) + bu_{xxy}(\tilde{\theta}). \end{aligned}$$

Observe that $\lambda := u_{xxx}(\tilde{\theta})$, $\mu := u_{xxy}(\tilde{\theta})$ and $\nu := u_{xyy}(\tilde{\theta})$ can be taken, thanks to (5.31), as coordinates on \mathcal{E}_{θ} , and as such their values can be arbitrary. Hence, $(\Sigma_{[1]}\mathcal{E})_{\theta}$ is the three-dimensional projective subvariety of $\mathbb{P}(\mathcal{C}_{\theta})$ given, in the domain of definition of the coordinates (b, f_0, f_1, f_2) , by the mixed implicit/parametric equations

$$(5.37) \quad 0 = \nu b^3 - 6\mu b^2 + 3\lambda b - 1,$$

$$(5.38) \quad f_0 = \nu + b(\mu^2 - \lambda\nu),$$

$$(5.39) \quad f_1 = \mu + b\nu,$$

$$(5.40) \quad f_2 = \lambda + b\mu,$$

where $\lambda, \mu, \nu \in \mathbb{R}$. Observe that the first equation corresponds to the innermost union appearing in (5.36), while the last three equations correspond to the outermost one. As announced, system above can be brought in a purely implicit form. Indeed, (5.38), (5.39) and (5.40) allow to express λ, μ, ν in terms of f_0, f_1, f_2 ,

$$\begin{aligned} \mu &= \frac{(f_0 + f_1 f_2)b - f_1}{f_1 b^2 + f_2 b - 1}, \\ \nu &= \frac{f_1^2 b - f_0}{f_1 b^2 + f_2 b - 1}, \\ \lambda &= f_2 - \frac{((f_0 + f_1 f_2)b - f_1)b}{f_1 b^2 + f_2 b - 1}. \end{aligned}$$

These expressions, substituted in (5.37), give the polynomial equation

$$1 + 8f_1 b^2 + f_1^2 b^4 + (3f_2 b - 6f_1 b^2 - 4)f_2 b - 10f_0 b^3 = 0$$

of the three-dimensional projective variety $(\Sigma_{[1]}\mathcal{E})_{\theta} \subseteq \mathbb{P}(\mathcal{C}_{\theta})$. Finally, performing the change of coordinates of Lemma 28 above, we get the 2nd order ODE

$$1 + 10y^6 v^2 + 24y^4 \ddot{v} + 108y^8 \ddot{v}^2 + 4(8 - 16y^3 v + 25y^2 \dot{v} + 54y^4 \ddot{v})y^2 \dot{v} - 2f_0(11 + 36y^4 \ddot{v})y^3 = 0,$$

which is by construction contact invariant of \mathcal{E} .

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